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CONVERGENCE OF A CONSTRAINED FINITE ELEMENT DISCRETIZATION OF THE MAXWELL KLEIN GORDON EQUATION *

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Abstract. As an example of a simple constrained geometric non-linear wave equation, we study a numerical approximation of the Maxwell Klein Gordon equation. We consider an existing constraint preserving semi-discrete scheme based on finite elements and prove its convergence in space dimension 2 for initial data of finite energy.

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1. Introduction

Non-linear wave equations are at the heart of basic physical models. Fundamental particles are best described by the quantum version of the Yang-Mills-Higgs equations (YMH) and gravitational fields satisfy Einstein's equations for general relativity (GR). For the former the unknown is a connection on a certain vectorbundle over space-time, whereas for the latter it is a pseudo-Riemannian metric. The equations can in both cases be derived from a variational principle involving a Lagrangian with a large gauge-group giving rise to constraints. Partial differential equations involving unknowns from differential geometry and stemming from a variational principle will be called *geometric wave equations*.

The well-posedness of equations with such a rich structure has recently been proved in Sobolev spaces of relatively low regularity. This is relevant both to physics and numerical analysis, since norms related to the energy are the most natural and are most easily incorporated into stability arguments for numerical schemes. For an introduction to the mathematics of geometric wave equations see [25]. Well posedness in the energy norm for the Yang-Mills equation was proved in [19]. For general relativity the issue is not completely resolved; progress is surveyed in [18].

Numerical models exist for both GR and YMH but little if any numerical analysis is available for them. The only geometric wave equation for which we are aware of convergence proofs is the wave map equation [3]. With the long-term goal of understanding numerical schemes for GR and YMH we propose to study in this paper the simplest equation in the YMH family, namely the Maxwell-Klein-Gordon (MKG) equation obtained with the gauge group U(1).

In the MKG equation the unknowns are the electromagnetic field, described by a vector potential and a scalar (complex) field. The scalar field gives rise to a current exciting the electromagnetic field whereas the vector potential enters the coefficients of the wave-equation satisfied by the scalar field. While the wave equation and

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the Maxwell equations are both linear, the coupling creates a non-linear evolution equation. It is also important that electric charge should be conserved, which gives a non-linear constraint on the flow.

In [10] we introduced a finite element method for the YM equations where the constraint is satisfied by a special application of Lagrange multipliers. In [7] we generalized the method so that it covers all equations in the YMH family, in particular the MKG equation. In this paper we shall prove convergence for this scheme in space dimension 2 with continuous time. The essential features of the scheme is that it preserves energy which gives control over the curl of the vector potential, whereas the constraint gives a weak control over the divergence. Together curl control and divergence control would imply H^1 control, if it weren't for the fact that the finite element space we use, namely Nédélec's edge elements, are not in H^1. Nevertheless we prove a discrete analogue of the Sobolev embedding valid for Nédélec's edge elements, in the spirit of Kikuchi's compactness result [17] using recently constructed commuting interpolators defined on rough functions. Together with Kato's inequality this gives us strong convergence in L^p spaces of the discrete solutions. A duality argument gives control of the Lagrange multiplier sufficient to conclude that the limit of the discrete solutions satisfies the continuous equations. The difficulties arising in space dimension 3 are pointed out.

The paper is organized as follows. In section 2, after setting up notations, we give some preliminary results. Although we will use these preliminaries in a space of dimension 2, we considered interesting to state them in arbitrary dimension. Section 3 is then dedicated to the exposition of the equation considered and the semi-discrete scheme used. The convergence of the approximate solution of MKG is then obtained in section 4.

2. Preliminary

2.1. **Notations**

We consider Ω a bounded simply connected domain in $\mathbb{R}^n$ ($n \in \mathbb{N}^*$), with $\mathcal{C}^1$ and connected boundary.

2.1.1. *Continuous spaces*

We define the usual spaces:

L^p spaces.

- $L^p(\Omega)$ the classical L^p space of real valued functions.
- We say that $\phi \in L^p(\Omega, \mathbb{C})$ if $\mathcal{R}e(\phi) \in L^p(\Omega)$ and $\mathcal{I}m(\phi) \in L^p(\Omega)$, where $\mathcal{R}e$ and $\mathcal{I}m$ are the real and imaginary parts of a complex number. We define $\mathcal{I}$ such that $\mathcal{I}(a + ib) = ib$, i.e. $\mathcal{I} = i\mathcal{I}m$.

Sobolev spaces.

- $H^s(\Omega)$, $H^s_0(\Omega)$, $s \in \mathbb{N}$, the classical Sobolev spaces of real valued functions. $\|.\|_{H^1(\Omega)}$ the norm associated and $|.|_1$ the semi-norm in $H^1(\Omega)$.
- $H^s(\Omega, \mathbb{C}) := \big\{\phi \in L^2(\Omega, \mathbb{C}) \,|\, \mathcal{R}e(\phi) \in H^s(\Omega) \text{ and } \mathcal{I}m(\phi) \in H^s(\Omega)\big\}$, $H^s_0(\Omega, \mathbb{C})$, $s \in \mathbb{N}$, the Sobolev spaces of complex valued functions.
- We note also $W^{s,p}(\Omega)$, $W^{s,p}(\Omega, \mathbb{C})$ and $W^{s,p}_0(\Omega, \mathbb{C})$ the classical Sobolev spaces of order s in $L^p(\Omega)$. $\|.\|_{W^{s,p}(\Omega)}$ the norm associated and $|.|_{s,p}$ the semi-norm in $W^{s,p}_0(\Omega)$. The same kind of notations holds for complex valued Sobolev spaces.
- $W^{-s,p}(\Omega)$ is defined as the dual space of $W^{s,p'}_0(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

Scalar product.

- $\cdot$ denotes the classical scalar product of vectors in $\mathbb{R}^n$, $|.|$ its norm,
- $\langle ., . \rangle$ is the real valued scalar product on $L^2(\Omega, \mathbb{C})$, $||.||$ the associated L^2 norm. $\langle ., . \rangle$ can also be viewed as the duality product between Sobolev spaces.

Vectorial spaces.

- $\mathbf{L}^2(\Omega)$ is the space of square integrable vector potentials, same kind of notations holds for $\mathbf{H}^1(\Omega)$, $\mathbf{H}^s(\Omega)$, $\mathbf{L}^p(\Omega, \mathbb{C})$...

- $\mathbf{H}(\mathrm{curl}, \Omega)$ the space of vector potentials in $\mathbb{R}^n$ considered as vector fields and one forms, with square integrable curl; the analogue space for the divergence will be noted $\mathbf{H}(\mathrm{div}, \Omega)$. For basic notions on $\mathbf{H}(\mathrm{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$, see [16,23].

- If $n \geq 2$, $\mathbf{H}_0(\mathrm{curl}, \Omega) := \{\mathbf{A} \in \mathbf{H}(\mathrm{curl}, \Omega)$ such that $\gamma_\tau \mathbf{A} = 0$ on $\partial\Omega\}$ where $\gamma_\tau \mathbf{A}$ is the tangential component of $\mathbf{A}$ on $\partial\Omega$
- $\mathbf{V} := \{v \in \mathbf{H}_0(\mathrm{curl}, \Omega) |\, \mathrm{div}\, v = 0 \text{ in } \Omega\}$.
- For $q \geq 1$, $\mathbf{H}_q(\mathrm{curl}, \Omega) := \mathbf{H}_0(\mathrm{curl}, \Omega) \cap \mathbf{L}^q(\Omega)$.

For simplicity of notation, if there is no ambiguity, we will use $H^1(\Omega)$ also for $H^1(\Omega, \mathbb{C})$(the same remark holds for other spaces). Otherwise, it will be specified.

2.1.2. *Time dependence*

For $I \subseteq [0, T]$, $\mathcal{C}(I; X)$ is the space of continuous functions from I to X. $\mathcal{C}(0, T; X)$ will denote $\mathcal{C}([0, T]; X)$. We define for $1 \leq p \leq +\infty$, the Bochner spaces $L^p(0, T; X)$ for X a Banach space as in [27].

Finally $\mathcal{C}_w(0, T; X)$ denotes the set of weakly continuous functions in time with value in X, i.e.:
$u \in \mathcal{C}_w(0, T; X)$ means that $t \mapsto \langle u(t), l\rangle$ is continuous on $[0, T]$, for all $l \in X'$, dual of X.

2.1.3. *Semi-discretization*

Let $(\mathcal{T}_h)$ be a quasi-uniform family of simplicial meshes of the space Ω such that its mesh size is h. Approximations of fields are based on finite dimensional spaces and Finite Elements.

In the sequel,

- $\mathbb{P}_1$ is the space of affine functions.
- $S_h := \left\{p_h \in H_0^1(\Omega) \quad | \quad p_h|_K \in \mathbb{P}_1, \quad \forall K \in \mathcal{T}_h\right\}$
- Y_h^0 is the space of purely imaginary piecewise affine and continuous scalar functions on Ω vanishing on the boundary $\partial\Omega$.
- $\mathbf{Y}_h^1$ is the space of purely imaginary Whitney-one forms on Ω ([4,23]) with tangential component vanishing on the boundary $\partial\Omega$.
  $\mathbf{Y}_h^1 \subset i\mathbf{H}_0(\mathrm{curl}, \Omega)$.
- Z_h^0 the space of complex piecewise affine and continuous scalar functions with null value on the boundary $\partial\Omega$.
- $X_h := \mathbf{Y}_h^1 \times Z_h^0$.

We define the space of discrete divergence free vectors :

- $\mathbf{V}_h := \left\{v_h \in \mathbf{Y}_h^1 \text{ such that } \langle v_h, \mathrm{grad}\, \beta_h\rangle = 0, \quad \forall \beta_h \in Y_h^0\right\}$

Remark 2.1.

$$\mathbf{V}_h \nsubseteq i\mathbf{V}$$

We suppose that all these spaces are fitted with the curved boundary (as in [15]).

Throughout the paper, we will use the notation C to refer to generic constants (independent of h).

2.2. **Preliminary results**

In this section we present some preliminaries. All is stated in arbitrary dimension and will be used in following sections in the particular case of a domain of $\mathbb{R}^2$. These results are either quite classical, either generalizations of classical results (from the L^2 case to the L^q case, or from time independent fields to time dependent ones).

In the following theorem we recall the Kato's inequality. $|.|$ denotes here the modulus.

Theorem 2.2. *[20] $n \in \mathbb{N}$. If $\mathbf{A} : \mathbb{R}^n \to i\mathbb{R}^n$ is in $L^2_{loc}(\mathbb{R}^n)$, $f \in L^2(\mathbb{R}^n, \mathbb{C})$ and $(D + \mathbf{A})f \in L^2(\mathbb{R}^n)$, then $|f|$, the modulus of f, is in $H^1(\mathbb{R}^n)$ and the diamagnetic inequality :*

$$|D|f|(x)| \leq |(D + \mathbf{A})f(x)|$$

holds pointwise for almost every $x \in \mathbb{R}^n$.

The second result we state in this section is the well known Helmholtz decomposition of fields in $\mathbf{H}(\mathrm{curl}, \Omega)$. We have a statement both in the continuous and in the discrete case. For the domain considered (simply connected, with connected boundary), we have *in the continuous case*:

For every $u \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, there exists a unique $\mathring{u} \in \mathbf{V}$ and $p \in H^1_0(\Omega)$ such that:

$$u = \mathring{u} + \mathrm{grad}\, p.$$

and *in the discrete case*:

For every $u_h \in \mathbf{Y}^1_h$, there exists a unique $\mathring{u}_h \in \mathbf{V}_h$ and $p_h \in Y^0_h$ such that:

$$u_h = \mathring{u}_h + \mathrm{grad}\, p_h.$$

These results can be found for example in [2, 23].

In the following many results will rely on Sobolev imbeddings which we recall below [1].

Proposition 2.3. *For all $q \in]1, n[$, $p \in [q, \frac{nq}{n-q}] \cap]\frac{n}{n-1}, +\infty[$, one has a continuous imbedding*

$$L^q(\Omega) \hookrightarrow W^{-1,p}(\Omega),$$

which is compact when $p < \frac{nq}{n-q}$.

The study of the convergence of the scheme rely on norm estimates in both time and space and on the possibility to extract strongly converging subsequences. Thus we need the characterization of compact sets in the time dependent case in spaces $L^\infty(0, T; B)$, if B is a Banach; this has been studied for example by J.Simon in [27]. The following theorem gives a sufficient condition to be a compact set in $L^\infty(0, T; B)$ if B is a Banach.

Theorem 2.4. *[27] Suppose that X, B, Y are Banach spaces such that $X \subset B \subset Y$ are continuous imbeddings with compact imbedding $X \hookrightarrow B$. Let F be a bounded set in $L^\infty(0, T; X)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^r(0, T; Y)$ for some $r > 1$. Then F is relatively compact in $\mathcal{C}(0, T; B)$.*

Next propositions (2.5 to 2.11) are some generalization of some classical L^2 results in the L^p case (like propositions 2.5, 2.8, 2.9, 2.11) and/or in the time dependent case (like proposition 2.7, 2.10).

The object of next two propositions is to establish an analogue of the usual Kikuchi compactness property, in L^q and including time dependency. The property is first proved for fields independent of time, then the general property concerns time dependent fields.
Denote 2^* the number such that $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ for $n \geq 2$ (with the convention that $2^* = +\infty$ for $n = 2$).

Proposition 2.5. *Let $1 \leq q \leq 2^*$ ($q < 2^*$ if $n = 2$).*
There exists $C > 0$ such that for all $v_h \in \mathbf{V}_h$,

$$\|v_h\|_{L^q(\Omega)} \leq C \|\mathrm{curl}\, v_h\|_{L^2(\Omega)}$$

Furthermore if $1 \leq q < 2^*$ *and* $\|\operatorname{curl} v_h\|_{L^2(\Omega)} \leq C$ *then there exists* $v \in \mathbf{V}$ *such that a subsequence of* v_h *strongly converges in* $\mathbf{L}^q(\Omega)$ *to* v.

Proof: Let q be as in the statement of the theorem. We first prove the estimation in the $L^q(\Omega)$ space. We denote by:

- P the L^2-orthogonal projection on the space of purely imaginary L^2 divergence free vectors. The kernel of P is $\operatorname{grad} H_0^1$ and this projection conserves the curl, i.e.

$$\operatorname{curl} \circ P = \operatorname{curl} \tag{2.1}$$

  Furthermore $P\mathbf{V}_h \subset i\mathbf{V}$. Here and subsequently, we will denote $P_{\mathbf{V}}$ the induced map from $\mathbf{H}_0(\operatorname{curl}, \Omega)$ to $i\mathbf{V}$.
- Q_h the projection on $\mathbf{Y}_h^1$ constructed in [11] (the ones constructed in [2, 24] are also suitable in this case) defined in $L^1(\Omega)$. It verifies the following property:

$$\text{If } \operatorname{curl} v = 0, \text{ then } \operatorname{curl} Q_h v = 0 \tag{2.2}$$

With the regularity of the domain Ω considered since $P_{\mathbf{V}} v_h \in \mathbf{V}$, $\operatorname{div} P_{\mathbf{V}} v_h$ and $\operatorname{curl} P_{\mathbf{V}} v_h$ are in $L^q(\Omega)$, we deduce that $P_{\mathbf{V}} v_h$ is in $W^{1,q}(\Omega)$ (using arguments of regularity of solution of elliptic problems in a smooth domain, see for example [26]). We get also the estimation:

$$|P_{\mathbf{V}} v_h|_{1,q} \leq C \|\operatorname{curl} P_{\mathbf{V}} v_h\|_{L^q(\Omega)}. \tag{2.3}$$

Remark 2.6. *We have* $\operatorname{curl} v_h = \operatorname{curl} P_{\mathbf{V}}\, v_h = \operatorname{curl} Q_h P_{\mathbf{V}} v_h$

Let us first estimate $\|v_h\|_{L^q(\Omega)}$.

By triangular inequality,

$$\|v_h\|_{L^q(\Omega)} \leq \|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} + \|Q_h P_{\mathbf{V}} v_h - P_{\mathbf{V}} v_h\|_{L^q(\Omega)} + \|P_{\mathbf{V}} v_h\|_{L^q(\Omega)}.$$

(a) We have by Bramble-Hilbert type estimates :

$$\|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch |P_{\mathbf{V}} v_h|_{1,q} \leq Ch \|\operatorname{curl} P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch \|\operatorname{curl} v_h\|_{L^q(\Omega)}$$

so that we can use the result (4.5.11) in [6] and obtain:

$$\text{if } q \geq 2,\ \|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch h^{\frac{n}{q} - \frac{n}{2}} \|\operatorname{curl} v_h\|_{L^2(\Omega)}$$

and

$$\text{if } 1 < q < 2,\ \|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch \|\operatorname{curl} v_h\|_{L^2(\Omega)}.$$

In conclusion

$$\|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch h^{\min(0, \frac{n}{q} - \frac{n}{2})} \|\operatorname{curl} v_h\|_{L^2(\Omega)} \tag{2.4}$$

(b) Furthermore

$$\|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq Ch^{\min(0, \frac{n}{q} - \frac{n}{2})} \|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^2(\Omega)}$$

But since $\operatorname{curl}(v_h - Q_h P_{\mathbf{V}} v_h) = 0$, we have that $v_h \perp v_h - Q_h P_{\mathbf{V}} v_h$, and $P_{\mathbf{V}} v_h \perp v_h - Q_h P_{\mathbf{V}} v_h$. So that

$$\|v_h - Q_h P_{\mathbf{V}} v_h\|^2_{L^2(\Omega)} \leq \|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^2(\Omega)} \times \|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^2(\Omega)}$$

Therefore

$$\|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq C h^{\min(0, \frac{n}{q} - \frac{n}{2})} \|P_{\mathbf{V}} v_h - Q_h P_{\mathbf{V}} v_h\|_{L^2(\Omega)}$$

By (2.4), one concludes that

$$\|v_h - Q_h P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq C h h^{min(0, \frac{n}{q} - \frac{n}{2})} \|\operatorname{curl} v_h\|_{L^2(\Omega)} \tag{2.5}$$

(c) By the Sobolev imbedding of $H^1(\Omega)$ in $L^q(\Omega)$, we have:

$$\|P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq C \|P_{\mathbf{V}} v_h\|_{H^1(\Omega)}$$

Friedrich's inequality yields then

$$\|P_{\mathbf{V}} v_h\|_{L^q(\Omega)} \leq C \|\operatorname{curl} v_h\|_{L^2(\Omega)} \tag{2.6}$$

(d) Since $1 + \frac{n}{q} - \frac{n}{2} \geq 0$, we combine (2.4), (2.5) and (2.6) to conclude:

$$\exists C > 0 \text{ such that } \forall v_h \in \mathbf{V}_h,$$

$$\|v_h\|_{L^q(\Omega)} \leq C \|\operatorname{curl} v_h\|_{L^2(\Omega)} \tag{2.7}$$

It remains to prove that a subsequence of v_h converges strongly in $L^q(\Omega)$, if $1 < q < 2^*$.

$P_{\mathbf{V}} v_h$ is bounded in $H^1(\Omega)$ so that we deduce strong convergence in $\mathbf{L}^q(\Omega)$ to extraction of a subsequence. Then since for $1 < q < 2^*$, $1 + \frac{n}{q} - \frac{n}{2} > 0$, from (2.4) and (2.5) we deduce that to a subsequence v_h converges in $\mathbf{L}^q(\Omega)$ and has the same limit than $P_{\mathbf{V}} v_h$ in $L^q(\Omega)$. This concludes the proof. □

Proposition 2.5 can be generalized to fields with a time dependency:

Proposition 2.7. *Let $1 < q \leq 2^*$, ($q < 2^*$ if $n = 2$). There exists $C > 0$ such that for all $v_h \in L^\infty(0, T; \mathbf{V}_h)$*

$$\|v_h\|_{L^\infty(0,T;L^q(\Omega))} \leq C \|\operatorname{curl} v_h\|_{L^\infty(0,T;L^2(\Omega))}$$

Furthermore if there exists $C > 0$ such that

$$\|\operatorname{curl} v_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

and

$$\|\dot{v}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

then for all $1 < q < 2^*$ there exists $v \in L^\infty(0, T; i\mathbf{V})$ such that a subsequence of v_h strongly converges in $L^\infty(0, T; \mathbf{L}^q(\Omega))$ to v.

Proof: As all inequalities from the proof of proposition 2.5 can be transported to time dependent fields, the only point which has to be clarified is that a subsequence of v_h has a limit in $L^\infty(0, T; \mathbf{L}^q(\Omega))$.

v_h bounded in $L^\infty(0, T; \mathbf{H}(\operatorname{curl}, \Omega))$ implies that $P_{\mathbf{V}} v_h$ is also bounded in $L^\infty(0, T; \mathbf{H}^1(\Omega))$. Moreover, $P_{\mathbf{V}}$ satisfies $\dot{\widehat{P_{\mathbf{V}} v_h}} = P_{\mathbf{V}} \dot{v}_h$ and therefore

$$\|\dot{\widehat{P_{\mathbf{V}} v_h}}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq \|\dot{v}_h\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C$$

Applying theorem 2.4, $P_{\mathbf{V}} v_h$ converges strongly (considering a subsequence) in $\mathbf{L}^2(\Omega)$. Then using inequalities (2.4) and (2.5) for time dependent fields, $\|v_h - P_{\mathbf{V}} v_h\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ converges to 0 as h tends to 0. Then using interpolation inequality, the convergence of v_h in $L^\infty(0,T;\mathbf{L}^2(\Omega))$, and the fact that $\|v_h\|_{L^\infty(0,T;L^q(\Omega))}$ is bounded completes the proof. □

Next proposition give some stability results for projection on Finite Elements spaces. This result will particularly be needed in section 4.5.

Proposition 2.8. *Let P_h^1 be the L^2 projection on $\mathbf{Y}_h^1$ and P_h^0 be the L^2 projection on Z_h^0. Then:*

(a) *P_h^1 is stable in $\mathbf{L}^p$, and from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}(\mathrm{curl},\Omega)$.*
(b) *P_h^0 is stable in $H^s(\Omega)$, for all $-1 \le s \le 1$.*

Proof:

(a) For P_h^1.
- Stability in $\mathbf{L}^p(\Omega)$.
  Using the result in [14] in the vectorial case, one deduces the stability in the L^p-norm.
- Stability in $\mathbf{H}(\mathrm{curl},\Omega)$.
  Let Q_h be the operator constructed in [11]. It is stable both in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}(\mathrm{curl},\Omega)$. Using also the inverse inequality between $\mathbf{H}(\mathrm{curl},\Omega)$ and $\mathbf{L}^2(\Omega)$, one has:

$$\|P_h^1 u\|_{\mathbf{H}(\mathrm{curl},\Omega)} \le \|Q_h P_h^1 u - Q_h u\|_{\mathbf{H}(\mathrm{curl},\Omega)} + \|Q_h u\|_{\mathbf{H}(\mathrm{curl},\Omega)} \tag{2.8}$$
$$\le Ch^{-1}\|Q_h P_h^1 u - Q_h u\|_{\mathbf{L}^2(\Omega)} + \|Q_h u\|_{\mathbf{H}(\mathrm{curl},\Omega)} \tag{2.9}$$
$$\le Ch^{-1}\|P_h^1 u - u\|_{\mathbf{L}^2(\Omega)} + \|Q_h u\|_{\mathbf{H}(\mathrm{curl},\Omega)} \tag{2.10}$$
$$\le C\|u\|_{\mathbf{H}^1(\Omega)} \tag{2.11}$$

(b) For P_h^0.
- The stability in $H^1(\Omega)$ comes from the result in [5,13]
- The stability in $H^{-1}(\Omega)$ follows by duality.
- The stability in $H^s(\Omega)$, for $-1 \le s \le 1$ is obtained by using interpolation inequalities.

□

Discretizing continuous equations leads to discrete ones which should have good convergence properties. Next proposition states this for a particular class of equations.

Proposition 2.9. *Let $p \in]1,+\infty[$ be given, $a(.,.)$ the bilinear form on $W^{1,p}(\Omega) \times W^{1,p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) given by:*

$$a(u,v) = \int_\Omega \mathrm{grad}\, u \cdot \mathrm{grad}\, v \, d\Omega$$

and for $h > 0$, $f_h \in W^{-1,p}(\Omega)$, $f \in W^{-1,p}(\Omega)$.
Let also $u_h \in Y_h^0$ be the solution of

$$a(u_h, v_h) = \langle f_h, v_h \rangle, \forall v_h \in Y_h^0$$

and $u \in iW_0^{1,p}(\Omega)$ the solution of

$$a(u,v) = \langle f, v \rangle, \forall v \in iW_0^{1,p'}(\Omega).$$

Then

(i) $\|u_h\|_{W^{1,p}(\Omega)} \le C\|f_h\|_{W^{-1,p}(\Omega)}$
(ii) *If $f_h \underset{h\to 0}{\longrightarrow} f$ in $W^{-1,p}(\Omega)$, then $u_h \underset{h\to 0}{\longrightarrow} u$ in $iW_0^{1,p}(\Omega)$.*

As a verifies an Inf-Sup condition on $W^{1,p}(\Omega)\times W^{1,p'}(\Omega)$ see [6,26], the result is deduced using Strang lemma (see [9,12]).
The following proposition is the corresponding generalization for time dependent fields:

Proposition 2.10. *Let $T>0$ and $p\in]1,+\infty[$ be given, $a(.,.)$ a bilinear form on $W^{1,p}(\Omega)\times W^{1,p'}(\Omega)$ ($\frac{1}{p}+\frac{1}{p'}=1$) given by:*

$$a(u,v)=\int_\Omega \operatorname{grad} u\cdot \operatorname{grad} v\, d\Omega$$

and for $h>0$, $f_h\in L^\infty(0,T;W^{-1,p}(\Omega))$, $f\in L^\infty(0,T;W^{-1,p}(\Omega))$.
Let also $u_h\in L^\infty(0,T;Y_h^0)$ be the solution of

$$a(u_h(t),v_h)=\langle f_h(t),v_h\rangle, \forall v_h\in Y_h^0, \text{ for a.e. } t \text{ in } [0,T]$$

and $u\in L^\infty(0,T;iW_0^{1,p}(\Omega))$ the solution of

$$a(u(t),v)=\langle f(t),v\rangle, \forall v\in iW_0^{1,p'}(\Omega) \text{ for a.e. } t \text{ in } [0,T].$$

Then

(i) $\|u_h\|_{L^\infty(0,T;W^{1,p}(\Omega))}\leq C\|f_h\|_{L^\infty(0,T;W^{-1,p}(\Omega))}$

(ii) *If $f_h \underset{h\to 0}{\longrightarrow} f$ in $L^\infty(0,T;W^{-1,p}(\Omega))$, then $u_h \underset{h\to 0}{\longrightarrow} u$ in $L^\infty(0,T;W_0^{1,p}(\Omega))$.*

Working with fields in $\mathbf{H}_q(\text{curl},\Omega)$ leads to a need of L^p stability of the Helmholtz decomposition as stated in the following

Proposition 2.11. *The Helmholtz decomposition in $\mathbf{Y}_h^1$ is stable in L^p-norm.*

Proof: Let $\mathbf{E}\in\mathbf{Y}_h^1$. From the L^2 Helmholtz decomposition, there exists $\mathring{\mathbf{E}}\in\mathbf{V}_h$ and $p_h\in Y_h^0$ such that $\mathbf{E}_h=\mathring{\mathbf{E}}_h+\operatorname{grad} p_h$. Keeping notations from proposition 2.9, we deduce that $\operatorname{div}\mathbf{E}_h\in W^{-1,p}(\Omega)$, and $a(p_h,v_h)=\langle \operatorname{div}\mathbf{E}_h,v_h\rangle, \forall v_h\in Y_h^0$. And so:

$$\|p_h\|_{W^{1,p}(\Omega)}\leq C\|\mathbf{E}_h\|_{L^p(\Omega)}$$

The stability of the decomposition follows. □

Finally we state a compact perturbation result (proposition 2.12) and a result on dual estimates (proposition 2.13) which we will use in section 4.5 to get estimates on the time derivative of the discrete solutions and on the Lagrange multiplier.

The following proposition is a generalization of the result obtained in [9] proposition A.5.2 (see also [8] theorem 1.12, corollary 1.17). Y^* denotes the topological dual space of Y.

Proposition 2.12. *Let X and Y two reflexive Banach spaces and $\mathcal{A}:X\to Y^*$ a continuous linear map with closed range. Let $\mathcal{K}$ denote a relatively compact set of compact operators $X\to Y^*$. Then, let (X_h) and (Y_h) be two families of subspaces of equal finite dimension of X and Y. Suppose that (Y_h) verifies an approximation property, $\mathcal{A}$ satisfies a discrete uniform inf-sup condition on $X_h\times Y_h$, and for all $\mathcal{B}\in\mathcal{K}$, $\mathcal{A}+\mathcal{B}$ is injective. Then there exists a constant C such that for all $\mathcal{B}\in\mathcal{K}$, $\mathcal{A}+\mathcal{B}$ verifies a uniform discrete inf-sup condition with constant C.*

Proof: We apply the result of [9]. For every $\mathcal{B}\in\mathcal{K}$, one can construct a ball $B(\mathcal{B},r_\mathcal{B})$ of center $\mathcal{B}$ and radius $r_\mathcal{B}$ sufficiently small, such that forall $\mathcal{B}'\in B(\mathcal{B},r_\mathcal{B})$, $\mathcal{A}+\mathcal{B}'$ verifies an inf-sup condition independent of $\mathcal{B}'$. Denote the corresponding constant by $C_\mathcal{B}$. Since $\{B(\mathcal{B},r_\mathcal{B}), \mathcal{B}\in\mathcal{K}\}$ covers $\mathcal{K}$, we can extract from it a finite subcover. Let C be the minimal corresponding inf-sup constant. Then for all $\mathcal{B}\in\mathcal{K}$, $\mathcal{A}+\mathcal{B}$ verifies a uniform inf-sup condition with constant C. This concludes the proof. □

Proposition 2.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces equipped with respectively the norms $\|.\|_{\mathcal{X}}$ and $\|.\|_{\mathcal{Y}}$, $a(.,.)$ a continuous bilinear form on $\mathcal{X} \times \mathcal{Y}$.*
Then let $\mathcal{X}_h$ and $\mathcal{Y}_h$ denote two families of subspaces of finite dimension of $\mathcal{X}$ and $\mathcal{Y}$ respectively. We suppose that $a(.,.)$ verifies discrete inf-sup conditions on $\mathcal{X}_h \times \mathcal{Y}_h$. We consider $T_h : \mathcal{Y}' \rightarrow \mathcal{X}_h$, such that for all $u \in \mathcal{Y}'$:

$$a(T_h u, v_h) = \langle u, v_h \rangle,\ \forall v_h \in \mathcal{Y}_h \tag{2.12}$$

and $T'_h : \mathcal{X}' \rightarrow \mathcal{Y}_h$, such that for all $v \in \mathcal{X}'$:

$$a(u_h, T'_h v) = \langle u_h, v \rangle, \forall u_h \in \mathcal{X}_h \tag{2.13}$$

Let $\mathcal{X}_+$ and $\mathcal{Y}_-$ be two other Banach spaces (with respective norms $\|.\|_{\mathcal{X}_+}$ and $\|.\|_{\mathcal{Y}_-}$) such that $\mathcal{X} \subset \mathcal{X}_+$ and $\mathcal{Y}_- \subset \mathcal{Y}$, and suppose that if $v \in \mathcal{X}'_+$ then $T'_h v \in \mathcal{Y}_-$ and one has

$$\|T'_h v\|_{\mathcal{Y}_-} \leq C \|v\|_{\mathcal{X}'_+}. \tag{2.14}$$

Then for all u in $\mathcal{Y}'_-$,

$$\|T_h u\|_{\mathcal{X}_+} \leq C \|u\|_{\mathcal{Y}'_-}$$

Proof: Existence of solutions are garenteed by inf-sup conditions. From (2.12), (2.13) and (2.14), we deduce:

$$\begin{aligned}
\|T_h u\|_{\mathcal{X}_+} &= \sup_{v \in \mathcal{X}'_+} \frac{\langle T_h u, v \rangle}{\|v\|_{\mathcal{X}'_+}} \\
&= \sup_{v \in \mathcal{X}'_+} \frac{a(T_h u, T'_h v)}{\|v\|_{\mathcal{X}'_+}} \\
&= \sup_{v \in \mathcal{X}'_+} \frac{\langle u, T'_h v \rangle}{\|v\|_{\mathcal{X}'_+}} \\
&\leq \frac{\|u\|_{\mathcal{Y}'_-} \|T'_h v\|_{\mathcal{Y}_-}}{\|v\|_{\mathcal{X}'_+}} \\
&\leq C \|u\|_{\mathcal{Y}'_-}
\end{aligned}$$

□

This proposition generalizes to the time dependent case in an obvious way.

All these preliminary results are valid in a space of arbitrary dimension. From now on we will consider a domain Ω included in $\mathbb{R}^2$, and study Maxwell-Klein-Gordon equation in this case. However all along the article the difficulties of the 3 dimensional case will be pointed out.

3. Equation and discrete formulation

Let $\Omega \subset \mathbb{R}^2$. $q \in \mathbb{N}^*$.

3.1. Continuous formulation

3.1.1. *General setting*

Let $T > 0$ be given. Solving the Maxwell Klein Gordon equation consists in finding :

- a time dependent gauge potential defined on $[0, T]$, $t \mapsto \mathcal{A}(t) = \begin{pmatrix} \alpha(t) \\ \mathbf{A}(t) \end{pmatrix}$ where $\alpha(t)$ is a purely imaginary scalar function on Ω and $\mathbf{A}(t)$ is a purely imaginary vector function on Ω

and

- a time dependent complex scalar function on Ω defined on $[0,T]$: $t \mapsto \phi(t)$

which are a critical point of *the action*:

$$\mathcal{S}(\mathbf{A},\phi,\alpha) = \frac{1}{2}\int_0^T -\|\mathrm{grad}\,\alpha - \dot{\mathbf{A}}\|^2_{L^2(\Omega)} + \|\mathrm{curl}\,\mathbf{A}\|^2_{L^2(\Omega)} - \|\dot{\phi} + \alpha\phi\|^2_{L^2(\Omega)} + \|D_{\mathbf{A}}\phi\|^2_{L^2(\Omega)}.$$

where $D_{\mathbf{A}}\phi = D\phi + \mathbf{A}\phi$ is the covariant derivative of ϕ.

We can express the variation of $\mathcal{S}$ at $(\alpha,\mathbf{A},\phi)$ in the direction $(\alpha',\mathbf{A}',\phi')$. Then the stationarity of *the action* gives Euler-Lagrange equations:

- Variation with respect to $\mathbf{A}'$ gives an evolution equation for $\mathbf{A}$
- Variation with respect to ϕ' gives an evolution equation for ϕ
- Variation with respect to α' gives a constraint

For more details on this we refer to [7].

3.1.2. *In the temporal gauge*

From now on we turn to Maxwell Klein Gordon equation in the temporal gauge, that is, we impose $\alpha(t) \equiv 0$. Equations are then given in $[0,T]$ by:

$$\begin{aligned}
\dot{\mathbf{A}} &= -\mathbf{E} && (3.1)\\
\dot{\phi} &= -\psi && (3.2)\\
\dot{\mathbf{E}} &= \mathrm{curl}(\mathrm{curl}(\mathbf{A})) + \mathcal{I}(D_{\mathbf{A}}\phi\bar{\phi}) && (3.3)\\
\dot{\psi} &= D^*_{\mathbf{A}}D_{\mathbf{A}}\phi && (3.4)
\end{aligned}$$

The constraint is given by:

$$\mathrm{div}(E) + \mathcal{I}(\psi\bar{\phi}) = 0 \tag{3.5}$$

We suppose that initial conditions are:

$$\begin{aligned}
\mathbf{A}(0,.) &= \mathbf{A}^0(.) \in i\mathbf{H}_0(\mathrm{curl},\Omega) \cap i\mathbf{H}^1(\Omega) && (3.6)\\
\phi(0,.) &= \phi^0(.) \in H^1_0(\Omega,\mathbb{C}) && (3.7)\\
\mathbf{E}(0,.) &= \mathbf{E}^0(.) \in i\mathbf{L}^2(\Omega) && (3.8)\\
\psi(0,.) &= \psi^0(.) \in L^2(\Omega,\mathbb{C}) && (3.9)
\end{aligned}$$

and that they verify the constraint given by (3.5) (in a weak sense in $H^{-1}(\Omega)$).
We define the energy of field at time t by:

$$\mathcal{H}(t) = \langle \mathbf{E},\mathbf{E}\rangle + \langle \mathrm{curl}\,\mathbf{A},\mathrm{curl}\,\mathbf{A}\rangle + \langle \psi,\psi\rangle + \langle D_{\mathbf{A}}\phi, D_{\mathbf{A}}\phi\rangle \tag{3.10}$$

and have that

$$\mathcal{H}(0) < +\infty \tag{3.11}$$

Proposition 3.1. *This energy is conserved in time for smooth solutions.*

For a proof see again [7] in a general setting.
In the rest of the paper we often drop the complex signs i and $\mathbb{C}$ (in the continuous case) for simplicity of notations.

Weak solution. We introduce the notion of weak solution to (3.1)-(3.4).

Definition 3.2. $(\mathbf{E}, \mathbf{A}, \psi, \phi)$ *is said to be a* ***weak solution*** *of (3.1)-(3.4), if*

- *There exists* $q > 2$, *such that*
  - $\mathbf{E} \in \mathcal{C}(0,T;\mathbf{H}^{-1}(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$,
  - $\mathbf{A} \in \mathcal{C}(0,T;\mathbf{L}^2(\Omega)) \cap L^\infty(0,T;\mathbf{H}_q(\mathrm{curl},\Omega) \cap \mathbf{H}^1(\Omega))$,
  - $\psi \in \mathcal{C}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$,
  - $\phi \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega))$
- $\left\{ \begin{array}{rcl} \dot{\mathbf{A}} & = & -\mathbf{E} \\ \dot{\phi} & = & -\psi \end{array} \right.$
- *For every* $(\mathbf{E}', \psi') \in \mathcal{C}^\infty_c(]0,T[\times\Omega)^2 \times \mathcal{C}^\infty_c(]0,T[\times\Omega)$, *there holds*

$$-\int_0^T \langle \mathbf{E}, \dot{\mathbf{E}}' \rangle dt - \int_0^T \langle \psi, \dot{\psi}' \rangle dt = \int_0^T \langle \mathrm{curl}\,\mathbf{A}, \mathrm{curl}\,\mathbf{E}' \rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi, \phi \mathbf{E}' \rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi, D_{\mathbf{A}}\psi' \rangle dt \qquad (3.12)$$

3.2. Discrete Formulation

3.2.1. *A saddle point problem*

Considering variational formulation of (3.3) and (3.4), and simply discretizing in space in X_h provides a scheme which violates the constraint (3.5). In order to preserve it we consider the following constraint preserving scheme (for details see [7]) arising as a saddle point problem:
For $T > 0$, find $(\mathbf{A}_h, \phi_h) \in X_h$ and a Lagrange multiplier $\beta_h \in Y_h^0$ such that for all $t \in [0,T]$:

$$\begin{array}{rcl} \dot{\mathbf{A}}_h & = & -\mathbf{E}_h \qquad (3.13) \\ \dot{\phi}_h & = & -\psi_h \qquad (3.14) \end{array}$$

$\forall (\mathbf{E}'_h, \psi'_h) \in X_h$,

$$\langle \dot{\mathbf{E}}_h, \mathbf{E}'_h \rangle + \langle \dot{\psi}_h, \psi'_h \rangle + \langle \mathbf{E}'_h, \mathrm{grad}\,\beta_h \rangle - \langle \psi'_h, \phi_h \beta_h \rangle = \langle \mathrm{curl}\,\mathbf{A}_h, \mathrm{curl}\,\mathbf{E}'_h \rangle + \langle D_{\mathbf{A}_h}\phi_h, \phi_h \mathbf{E}'_h \rangle + \langle D_{\mathbf{A}_h}\phi_h, D_{\mathbf{A}_h}\psi'_h \rangle, \qquad (3.15)$$

$$\langle \dot{\mathbf{E}}_h, \mathrm{grad}\,\beta'_h \rangle - \langle \dot{\psi}_h, \phi_h \beta'_h \rangle = 0, \ \forall \beta'_h \in Y_h^0 \qquad (3.16)$$

with initial conditions:

$$\begin{array}{rcl} \mathbf{A}_h(0,.) & = & \mathbf{A}_h^0 \in \mathbf{Y}_h^1 \qquad (3.17) \\ \mathbf{E}_h(0,.) & = & \mathbf{E}_h^0 \in \mathbf{Y}_h^1 \qquad (3.18) \\ \phi_h(0,.) & = & \phi_h^0 \in Z_h^0 \qquad (3.19) \\ \psi_h(0,.) & = & \psi_h^0 \in Z_h^0 \qquad (3.20) \end{array}$$

where we suppose that $\mathbf{A}_h^0$, $\mathbf{E}_h^0$, ϕ_h^0, ψ_h^0 are chosen such that:

- $\mathbf{A}_h^0 \underset{h\to 0}{\longrightarrow} \mathbf{A}^0$ in $\mathbf{H}_q(\mathrm{curl},\Omega)$, $\forall q < +\infty$,
- $\mathbf{E}_h^0 \underset{h\to 0}{\longrightarrow} \mathbf{E}^0$ in $\mathbf{L}^2(\Omega)$,
- $\phi_h^0 \underset{h\to 0}{\longrightarrow} \phi^0$ in $L^2(\Omega) \cap L^q(\Omega)$, $\forall q < +\infty$,
- $\psi_h^0 \underset{h\to 0}{\longrightarrow} \psi^0$ in $L^2(\Omega)$

3.2.2. *Existence of a solution to the discrete formulation*

The above equation can be viewed with $(\mathbf{A}_h, \phi_h) \in X_h$ as given parameters. We then can rewrite equations (3.15) and (3.16) as:

$$\begin{aligned} \langle \dot{\mathbf{E}}_h, \mathbf{E}'_h\rangle + \langle \dot{\psi}_h, \psi'_h\rangle + \langle \mathbf{E}'_h, \operatorname{grad}\beta_h\rangle - \langle \psi'_h, \phi_h\beta_h\rangle &= f_{\mathbf{A}_h,\phi_h}(\mathbf{E}'_h) + g_{\mathbf{A}_h,\phi_h}(\psi'_h) && (3.21)\\ \langle \dot{\mathbf{E}}_h, \operatorname{grad}\beta'_h\rangle - \langle \dot{\psi}_h, \phi_h\beta'_h\rangle &= 0 && (3.22)\end{aligned}$$

where

$$f_{\mathbf{A}_h,\phi_h}(\mathbf{E}'_h) = \langle \operatorname{curl}\mathbf{A}_h, \operatorname{curl}\mathbf{E}'_h\rangle + \langle D_{\mathbf{A}_h}\phi_h, \phi_h\mathbf{E}'_h\rangle$$

and

$$g_{\mathbf{A}_h,\phi_h}(\psi'_h) = \langle D_{\mathbf{A}_h}\phi_h, D_{\mathbf{A}_h}\psi'_h\rangle$$

Proposition 3.3. *Let $h > 0$ be fixed. The system given by (3.21) and (3.22) with unknowns $(\dot{\mathbf{E}}_h, \dot{\psi}_h, \beta_h)$ has a unique solution in $X_h \times Y_h^0$. Furthermore, the solution depends smoothly on the parameters (A_h, ϕ_h).*

Taking $\mathbf{E}'_h = \operatorname{grad}\beta'_h$ and $\psi'_h = 0$ gives the following discrete Babuska-Brezzi compatibility condition:

$$\inf_{\beta'_h\in Y_h^0}\ \sup_{(\mathbf{E}'_h,\psi'_h)\in X_h} \frac{\langle \mathbf{E}'_h, \operatorname{grad}\beta'_h\rangle - \langle \psi'_h, \phi_h\beta'_h\rangle}{(|\beta'_h|^2 + |\operatorname{grad}\beta'_h|^2)^{\frac12}(|\mathbf{E}'_h|^2 + |\psi'_h|^2)^{\frac12}} \geq \frac{1}{C}$$

where C is a positive constant independent of the time t and of ϕ_h. Since h is fixed and all spaces we are dealing with are of finite dimension and all the considered operators are polynomial in the unknowns, we have proved the proposition. $\square$

We denote $\mathcal{P}_{\mathbf{Y}_h^1}$ the projection from $X_h \times Y_h^0$ on $\mathbf{Y}_h^1$, and $\mathcal{P}_{Z_h^0}$ the projection from $X_h \times Y_h^0$ on Z_h^0.

If $\mathcal{S}$ is the solution operator associated to equation (3.21), we are able to solve in X_h the equations

$$\begin{aligned} \ddot{\mathbf{A}}_h &= -\mathcal{P}_{\mathbf{Y}_h^1}\circ\mathcal{S}(\mathbf{A}_h, \phi_h) && (3.23)\\ \ddot{\phi}_h &= -\mathcal{P}_{Z_h^0}\circ\mathcal{S}(\mathbf{A}_h, \phi_h) && (3.24)\end{aligned}$$

locally in time with initial conditions giving by (3.6), (3.7), (3.8), (3.9).
Conclusion: We have existence of $(A_h, \phi_h) \in X_h$ for the discrete formulation locally in time.
We define the discrete energy at any time:

$$\mathcal{H}_h(t) = \frac12\left(\langle \mathbf{E}_h, \mathbf{E}_h\rangle(t) + \langle \operatorname{curl}\mathbf{A}_h, \operatorname{curl}\mathbf{A}_h\rangle(t) + \langle \psi_h, \psi_h\rangle(t) + \langle D_{\mathbf{A}_h}\phi_h, D_{\mathbf{A}_h}\phi_h\rangle(t)\right)$$

From approximation of initial conditions:

$$\mathcal{H}_h^0 = \frac12\left(\langle \mathbf{E}_h^0, \mathbf{E}_h^0\rangle + \langle \operatorname{curl}\mathbf{A}_h^0, \operatorname{curl}\mathbf{A}_h^0\rangle + \langle \psi_h^0, \psi_h^0\rangle + \langle D_{\mathbf{A}_h^0}\phi_h^0, D_{\mathbf{A}_h^0}\phi_h^0\rangle\right) < +\infty \qquad (3.25)$$

One can find in [7] a detailed proof of the following:

Proposition 3.4. *Equations (3.13)-(3.16) preserve the constraint and the energy of the system.*

This implies that $\mathbf{E}_h$ and ψ_h are bounded in $L^\infty(0, T, L^2(\Omega))$, and so are $\mathbf{A}_h$ and ϕ_h so that they are defined in the whole $[0, T]$.
Conclusion: We have global existence in time of the solution of equations (3.13)-(3.20).

We now would like to prove that the sequence $(\mathbf{E}_h,\ \mathbf{A}_h,\ \psi_h, \phi_h)$ converges (in a sense which has to be made precise) to a weak solution of the Maxwell Klein Gordon equation (in the sense of definition 3.2).

4. Convergence of the solution

The rest of the paper is dedicated to prove the following result:

Theorem 4.1. *Let $\mathbf{A}^0$, ϕ^0, $\mathbf{E}^0$, ψ^0 given as in (3.6)-(3.9). There exists*
- $\mathbf{E} \in \mathcal{C}(0,T;\mathbf{H}^{-1}(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$
- $\mathbf{A} \in \mathcal{C}(0,T;\mathbf{L}^2(\Omega)) \cap L^\infty(0,T;\mathbf{H}_q(\mathrm{curl},\Omega) \cap \mathbf{H}^1(\Omega))$
- $\psi \in \mathcal{C}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$
- $\phi \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega))$

such that the sequence $(\mathbf{E}_h,\mathbf{A}_h,\psi_h,\phi_h)$ solution of (3.13)-(3.20) converges to $(\mathbf{E},\mathbf{A},\psi,\phi)$ with
- $\mathbf{E}_h \underset{h\to 0}{\longrightarrow} \mathbf{E}$ *in* $\mathcal{C}(0,T;\mathbf{H}^{-1}(\Omega))$, $\mathbf{E}_h \underset{h\to 0}{\rightharpoonup} \mathbf{E}$ *in* $L^\infty(0,T;\mathbf{L}^2(\Omega))$ *weak-star,*
- $\mathbf{A}_h \underset{h\to 0}{\longrightarrow} \mathbf{A}$ *in* $\mathcal{C}(0,T;\mathbf{L}^q(\Omega))$, $\forall 1<q<+\infty$, $\mathrm{curl}\,\mathbf{A}_h \underset{h\to 0}{\rightharpoonup} \mathrm{curl}\,\mathbf{A}$ *in* $L^\infty(0,T;\mathbf{L}^2(\Omega))$ *weak-star,*
- $\psi_h \underset{h\to 0}{\longrightarrow} \psi$ *in* $\mathcal{C}(0,T;H^{-1}(\Omega))$, $\psi_h \underset{h\to 0}{\rightharpoonup} \psi$ *in* $L^\infty(0,T;L^2(\Omega))$ *weak-star,*
- $\phi_h \underset{h\to 0}{\longrightarrow} \phi$ *in* $\mathcal{C}(0,T;L^q(\Omega))$, $\forall 1<q<+\infty$, $\mathrm{grad}\,\phi_h \underset{h\to 0}{\rightharpoonup} \mathrm{grad}\,\phi$ *in* $L^\infty(0,T;L^2(\Omega))$ *weak-star.*

Furthermore $(\mathbf{E},\mathbf{A},\psi,\phi)$ is a weak solution of the Maxwell Klein Gordon equation given by (3.1)-(3.9) in the sense of definition 3.2 with initial conditions given by (3.6)-(3.9).

Looking at the right hand side of (3.12) points out that the convergence of the non linear terms is subjected to strong convergence in L^q spaces of either ϕ_h or $\mathbf{A}_h$. This explains the type of convergence obtained in theorem 4. Some convergence can be deduced from a priori estimates (mostly weak convergence). In order to be able to conclude to strong convergence we will use compacity arguments.
We first conclude to strong convergence for ϕ_h thanks to a priori estimates on $\mathbf{A}_h$ (sections 4.1.2 and 4.2). Then section 4.3 is dedicated to strong convergence on the gauge potential $\mathbf{A}_h$. Finally section 4.5 leads to strong convergence for $\mathbf{E}_h$, ψ_h and weak-* convergence on the Lagrange multiplier β_h.

4.1. A priori estimates

4.1.1. *Bounds in the energy norm*

From energy conservation (see 3.2), we deduce the following bounds

$$||\mathbf{E}_h||_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C \tag{4.1}$$
$$||\mathrm{curl}\,\mathbf{A}_h||_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C \tag{4.2}$$
$$||\psi_h||_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{4.3}$$
$$||D_{\mathbf{A}_h}\phi_h||_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C \tag{4.4}$$

Kato's inequality, Theorem 2.2 gives:

$$\|D\,|\phi_h|\,\|_{\mathbf{L}^2(\Omega)} \leq C, \text{ a.e. in } [0,T].$$

And so
$$\|\,|\phi_h|\,\|_{L^\infty(0,T;H^1_0(\Omega))} \leq C \tag{4.5}$$
Using Sobolev imbedding, $\forall 1 \leq q < +\infty$:

$$\|\phi_h\|_{L^\infty(0,T;L^q(\Omega))} \leq C \tag{4.6}$$

We deduce some weak convergence properties.

(a) It follows from (4.6) that there exists $\phi \in L^\infty(0,T;L^q(\Omega))$ such that

$$\phi_h \rightharpoonup \phi \text{ as } h \to 0 \text{ in } L^\infty(0,T;L^q(\Omega)) \text{ weak-*}, \forall q < +\infty. \tag{4.7}$$

By (4.3), we have also

$$\dot{\phi}_h \rightharpoonup \dot{\phi} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-*}. \tag{4.8}$$

(b) In a same way, from (4.1) we deduce weak-* convergence for $\dot{\mathbf{A}}_h$:

$$\dot{\mathbf{A}}_h \rightharpoonup \dot{\mathbf{A}} \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak-*}. \tag{4.9}$$

As well, using (4.2), we deduce:

$$\operatorname{curl} \mathbf{A}_h \rightharpoonup \operatorname{curl} \mathbf{A} \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak-*}. \tag{4.10}$$

Before concluding to strong convergence of ϕ_h, the first step is to extract some uniform estimates on $\mathbf{A}_h$ in the L^q norm, with $q < +\infty$.

4.1.2. *Estimate for the gauge potential* $\mathbf{A}_h$

The idea here is to exploit the discrete Helmholtz decomposition of $\mathbf{A}_h$ and give uniform estimates for each part of its decomposition.
Uniform estimates on the curl of $\mathbf{A}_h$ give estimates on the discrete divergence-free part of the gauge potential. In order to have estimates on $\mathbf{A}_h$, we will use the constraint which gives a bound on the divergence of $\mathbf{A}_h$. These estimates will be established in L^q for all $q < +\infty$, and also used later in the paper in the study of the convergence of the scheme.
Discrete Helmholtz decomposition. $\mathbf{A}_h$ can be uniquely decomposed as the sum of two orthogonal fields (see section 2.2):

$$\mathbf{A}_h(t) = \mathring{\mathbf{A}}_h(t) + \operatorname{grad} p_h\,(t)$$

where $\mathring{\mathbf{A}}_h(t) \in \mathbf{V}_h$, and $p_h(t) \in Y_h^0$ for almost all $t \in [0,T]$.
Estimates on discrete divergence free part. To obtain a uniform estimate in the L^q norm in space (for all $1 < q < +\infty$) of the discrete divergence free part in terms of the L^2 norm in space of the curl of the gauge potential $\mathbf{A}_h$, we apply the estimate of Theorem 2.7 to $\mathring{\mathbf{A}}_h$:
Then there holds

$$\|\mathring{\mathbf{A}}_h\|_{L^\infty(0,T;\mathbf{L}^q(\Omega))} \le C, \ \forall q < +\infty \tag{4.11}$$

Remark 4.2. *In 3D, we obtain* $\|\mathring{\mathbf{A}}_h\|_{L^\infty(0,T;\mathbf{L}^q(\Omega))} \le C$, $\forall 1 < q \le 6$

Estimates on the gradient part. The expression of the constraint, if verified at $t = 0$, gives us that: $\forall \beta'_h \in Y_h^0$,

$$\langle \dot{\mathbf{A}}_h(t), \operatorname{grad} \beta'_h \rangle = \langle \dot{\phi}_h(t), \phi_h(t)\beta'_h \rangle \text{ for almost every } t \in [0,T].$$

Integrating once more, we obtain

$$\langle \mathbf{A}_h(t), \operatorname{grad} \beta'_h \rangle = \langle \mathbf{A}_h(0), \operatorname{grad} \beta'_h \rangle + \int_0^t \langle \dot{\phi}_h, \phi_h \beta'_h \rangle$$

Using the discrete Helmholtz decomposition, we deduce that

$$\langle \operatorname{grad} p_h(t), \operatorname{grad} \beta'_h \rangle = \langle f_h(t), \beta'_h \rangle$$

where

$$f_h(t) = \operatorname{div} \mathbf{A}_h^0 + \int_0^t \dot{\phi}_h \bar{\phi}_h$$

Lemma 4.3. *$f_h \in L^\infty(0,T,W^{-1,q}(\Omega))$ for all $q < +\infty$ and is bounded independently of h in $L^\infty(0,T;W^{-1,q}(\Omega))$.*

Proof: Let $q < +\infty$ given.

(a) $\mathrm{div}\mathbf{A}_h^0$ is bounded in $W^{-1,q}(\Omega)$ by construction of $\mathbf{A}_h^0$.

(b) By section 4.1.1, $\dot\phi_h$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and $|\phi_h|$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, so we have :

$\dot\phi_h|\phi_h|$ is bounded in $L^\infty(0,T;L^r(\Omega))$ for all $r < 2$

Remark 4.4. *In the 3D case $\dot\phi_h|\phi_h| \in L^r(\Omega)$ with $1 < r \leq \frac{3}{2}$*

Let $u_h = \int_0^t \dot\phi_h \bar\phi_h dt$, we have $u_h \in L^\infty(0,T;L^r(\Omega))$ for all $r < 2$, and $\|u_h\|_{L^\infty(0,T;L^r(\Omega))} \leq C$. Using Sobolev imbedding 2.3, we deduce:

$$u_h \in L^\infty(0,T;W^{-1,q}(\Omega)) \text{ for all } q < +\infty$$

(a) and (b) allow then to conclude that

$$\|f_h\|_{L^\infty(0,T;W^{-1,q}(\Omega))} \text{ is bounded independently of } h \text{ for all } q < +\infty.$$

□

Remark 4.5. *In 3 dimensions, $f_h \in L^\infty(0,T;W^{-1,q}(\Omega))$ where $q \leq 3$.*

From proposition 2.9 and previous lemma, we deduce that for all $q < +\infty$ there exists $C(q)$ such that:

$$\|p_h\|_{L^\infty(0,T,W^{1,q}(\Omega))} \leq C(q)$$

Remark 4.6. *In the 3D case, we have:*

$$\|p_h\|_{L^\infty(0,T,W^{1,q}(\Omega))} \leq C(q) \text{ for } q \leq 3.$$

4.1.3. *Conclusion*

To conclude 4.1.1, and 4.1.2:

$$\|\mathbf{A}_h\|_{L^\infty(0,T;\mathbf{L}^q(\Omega))} \leq C(q),\ \forall\ q < +\infty \tag{4.12}$$

Remark 4.7. *In the 3D case,*

$$\forall\ q \leq 3,\ \|\mathbf{A}_h\|_{L^\infty(0,T;\mathbf{L}^q(\Omega))} \leq C(q) \tag{4.13}$$

4.2. Strong convergence on ϕ_h

We are able to conclude to strong convergence of ϕ_h in $\mathcal{C}(0,T;L^q(\Omega))$ for all $q < +\infty$.

We recall that : $D_{\mathbf{A}_h}\phi_h = D\phi_h + \mathbf{A}_h\phi_h$.

So

$$\|D\phi_h\|_{L^\infty(0,T;L^2(\Omega))} \leq \|D_{\mathbf{A}_h}\phi_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{A}_h\phi_h\|_{L^\infty(0,T;L^2(\Omega))}$$

From the estimates (4.12) and (4.6), we can now state :

$$\|\phi_h\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C$$

Remark 4.8. *To have this in 3 dimensions, we have to use that $A_h \in L^\infty(0,T;L^3(\Omega))$ and $\phi_h \in L^\infty(0,T;L^6(\Omega))$ which is the limit case.*

And as

$$\|\dot{\phi}_h\|_{L^r(0,T;L^2(\Omega))} \leq C, \text{ for } r > 1$$

We deduce by theorem 2.4 that:

$$\phi_h \to \phi \text{ as } h \to 0 \text{ in } \mathcal{C}(0,T;L^2(\Omega)) \tag{4.14}$$

Then as for all $q < +\infty$, ϕ_h is bounded in $L^\infty(0,T;L^q(\Omega))$ independently of t and h, we deduce by interpolation inequality that to a subsequence:

$$\phi_h \to \phi \text{ as } h \to 0 \text{ in } \mathcal{C}(0,T;L^q(\Omega)) \tag{4.15}$$

Remark 4.9. *In dimension 3, to extraction of a subsequence: $\phi_h \to \phi$ as $h \to 0$ in $L^\infty(0,T;L^q(\Omega))$ for all $q < 6$;*

In order to be able to pass to the limit in equations (3.13)-(3.16), we also need a strong convergence on $\mathbf{A}_h$. To do so, we use the discrete Helmholtz decomposition as before and deduce strong convergence separately on the discrete divergence free part and the gradient part.

4.3. Strong convergence on the gauge potential

4.3.1. *Strong convergence of the discrete divergence free part*

We know from energy estimates that:

$$\|\mathrm{curl}\, \mathring{\mathbf{A}}_h\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C$$

and from section 4.1.2:

$$\|\mathring{\mathbf{A}}_h\|_{L^\infty(0,T;\mathbf{L}^q(\Omega))} \leq C.$$

Since time derivation preserves discrete Helmholtz decomposition, we obtain:

$$\left\|\frac{\partial}{\partial t}\mathring{\mathbf{A}}_h\right\|_{L^r(0,T;\mathbf{L}^2(\Omega))} \leq C, \text{ for } r > 1$$

Therefore, we can apply proposition 2.7 to $\mathring{\mathbf{A}}_h$ and conclude:

There exists $\mathring{\mathbf{A}} \in L^\infty(0,T;i\mathbf{V})$ such that to extraction of a subsequence

$$\forall\ 1 \leq q < +\infty,\ \mathring{\mathbf{A}}_h \underset{h\to 0}{\longrightarrow} \mathring{\mathbf{A}} \text{ in } \mathcal{C}(0,T;L^q(\Omega)) \tag{4.16}$$

Remark 4.10. *In 3 dimensions to extraction of a subsequence, $\mathring{\mathbf{A}}_h \underset{h\to 0}{\longrightarrow} \mathring{\mathbf{A}}$ in $L^\infty(0,T;\mathbf{L}^q(\Omega))$, for $q < 3$.*

4.3.2. *Strong convergence on the gradient part*

We shall now derive strong convergence for $\mathrm{grad}\, p_h$ appearing in the discrete Helmholtz decomposition. We keep notations of paragraph 4.1.2.
Let $f = \mathrm{div}(\mathbf{A}^0) + \int_0^t \dot{\phi}\bar{\phi}$, we have $f \in L^\infty(0,T;W^{-1,q}(\Omega))$, for $q < +\infty$.
We recall that

$$u_h = \int_0^t \dot{\phi}_h \bar{\phi}_h dt,$$

$$\|u_h\|_{L^\infty(0,T;L^r(\Omega))} \leq C \text{ for } 1 \leq r < 2.$$

and as $\dot{\phi}_h\bar{\phi}_h \in L^\infty(0,T;L^r(\Omega))$, $r < 2$, bounded independently of h, we deduce from theorem 2.3:

$$\|\dot{u}_h\|_{L^\infty(0,T;W^{-1,q}(\Omega))} \leq C.$$

From proposition 2.3, we deduce that there exists $u \in L^\infty(0,T;W^{-1,q}(\Omega))$ such that we can extract a subsequence still denoted u_h that converges to u in $\mathcal{C}(0,T;W^{-1,q}(\Omega))$, $\forall q < +\infty$. Furthermore, from weak convergence of $\dot{\phi}_h$ and strong convergence of ϕ_h in $L^\infty(0,T;L^2(\Omega))$, we deduce that $u = \int_0^t \dot{\phi}\bar{\phi}dt$ a.e. in $[0,T]$.

Remark 4.11. *In the 3D case, $u_h \to u$ in $L^\infty(0,T;W^{-1,q}(\Omega))$, $\forall q < 3$.*

Since $\mathbf{A}_h^0 \underset{h\to 0}{\longrightarrow} \mathbf{A}^0$ in $L^q(\Omega)$, $\forall\, q < +\infty$, we deduce also that to extraction of a subsequence

$$f_h \underset{h\to 0}{\longrightarrow} f \text{ in } L^\infty(0,T;W^{-1,q}(\Omega)),\ \forall\, q < +\infty.$$

Applying proposition 2.9 yields:
There exists $p \in L^\infty(0,T;iW_0^{1,q}(\Omega))$ such that $p_h \underset{h\to 0}{\longrightarrow} p$ in $L^\infty(0,T;W^{1,q}(\Omega))$.

4.3.3. *Strong convergence*

The decomposition $\mathbf{A}_h = \mathring{\mathbf{A}}_h + \operatorname{grad} p_h$ and the last two sections yield to extraction of a subsequence:

$$\mathbf{A}_h \underset{h\to 0}{\longrightarrow} \mathbf{A} := \mathring{\mathbf{A}} + \operatorname{grad} p \text{ in } L^\infty(0,T;\mathbf{L}^q(\Omega)) \text{ for all } q < +\infty. \tag{4.17}$$

Remark 4.12. *In 3 dimension: $\mathbf{A}_h \underset{h\to 0}{\longrightarrow} \mathbf{A}$ in $L^\infty(0,T;\mathbf{L}^q(\Omega))$ for all $q < 3$.*

4.4. **Conclusion**

To sum up, to extraction of a subsequence:

$$\mathbf{A}_h \underset{h\to 0}{\longrightarrow} \mathbf{A} \text{ in } \mathcal{C}(0,T;\mathbf{L}^q(\Omega)) \text{ for all } q < +\infty$$

$$\phi_h \underset{h\to 0}{\longrightarrow} \phi \text{ in } \mathcal{C}(0,T;L^q(\Omega)) \text{ for all } q < +\infty$$

We then have strong convergence in L^q spaces for both ϕ_h and $\mathbf{A}_h$. We are now looking at some estimation on the second time derivatives of these fields and on the Lagrange multiplier.

4.5. **Estimation by compact perturbation**

Let $q \in]1,+\infty[$ given, and $\mathcal{X}^q = \mathbf{L}^q(\Omega) \times L^2(\Omega) \times W^{1,q}(\Omega)$ equipped with the canonical norm.
Let $\mathbf{a}$ be the bilinear form given by:

$$\mathbf{a}(\mathbf{E},\psi,\beta;\mathbf{E}',\psi',\beta') = \langle \mathbf{E},\mathbf{E}'\rangle + \langle \psi,\psi'\rangle + \langle \mathbf{E}',\operatorname{grad}\beta\rangle + \langle \mathbf{E},\operatorname{grad}\beta'\rangle$$

and $\mathbf{b}_\phi$ the one given by:

$$\mathbf{b}_\phi(\mathbf{E},\psi,\beta;\mathbf{E}',\psi',\beta') = -\langle \psi',\phi\beta\rangle - \langle \psi,\phi\beta'\rangle$$

In the discrete setting, we will note $\mathcal{X}_h^q$ the space $X_h \times Y_h^0$ equipped with the $L^q \times L^2 \times W^{1,q}$-norm.
Finally q' is such that: $\frac{1}{q'} + \frac{1}{q} = 1$.

4.5.1. *Estimates*

This section is dedicated to the proof of the following:

Proposition 4.13. $q > 2$. *Let $(\dot{E}_h, \dot{\psi}_h, \beta_h) \in X_h \times Y_h^0$ be the solution of (3.13)-(3.20), so that*

$$(\mathbf{a} + \mathbf{b}_{\phi_h})(\dot{\mathbf{E}}_h, \dot{\psi}_h, \beta_h; \mathbf{E}_h', \psi_h', \beta_h') = f_{\mathbf{A}_h,\phi_h}(\mathbf{E}_h') + g_{\mathbf{A}_h,\phi_h}(\psi_h').$$

Then $\dot{\mathbf{E}}_h \in L^\infty(0,T;\mathbf{H}^{-1}(\Omega))$, $\dot{\psi}_h \in L^\infty(0,T;H^{-1}(\Omega))$, $\beta_h \in L^\infty(0,T;W^{1,q'}(\Omega))$ *with uniform bounds:*

$$\|\dot{\mathbf{E}}_h\|_{L^\infty(0,T;\mathbf{H}^{-1}(\Omega))} \le C,$$

$$\|\dot{\psi}_h\|_{L^\infty(0,T;H^{-1}(\Omega))} \le C$$

and

$$\|\beta_h\|_{L^\infty(0,T;W^{1,q'}(\Omega))} \le C$$

Proof: Let $q > 2$. $\mathbf{a}$ is continuous on $\mathcal{X}^q \times \mathcal{X}^{q'}$ and verifies a uniform discrete inf-sup condition on $\mathcal{X}_h^q \times \mathcal{X}_h^{q'}$ norm. This fact is proved in the following two steps:

(a) We have a discrete inf-sup condition on $(\beta_h, \mathbf{E}'_h) \mapsto \langle \mathbf{E}'_h, \operatorname{grad} \beta_h \rangle$:
Indeed grad maps Y_h^0 to $\mathbf{Y}_h^1$, so we deduce by [6] (chapter 8) or [26], and the stability of the L^2 projection in L^q (proposition 2.8), that:

$$\inf_{\beta_h \in Y_h^0} \sup_{\mathbf{E}'_h \in \mathbf{Y}_h^1} \frac{\langle \mathbf{E}'_h, \operatorname{grad} \beta_h \rangle}{\|\beta_h\|_{W^{1,q}(\Omega)} \|\mathbf{E}'_h\|_{\mathbf{L}^{q'}(\Omega)}} \ge \frac{1}{C} > 0 \tag{4.18}$$

The symmetric inequality (with q and q' exchanged) holds also.

(b) We have also a discrete inf-sup condition on the associated kernel:
Indeed as the L^2 projection is stable in the $L^q(\Omega)$-norm (proposition 2.8), and Helmholtz decomposition is stable in the L^q-norm (proposition 2.11), we deduce:

$$\forall \mathbf{E}_h \in \mathbf{V}_h, \sup_{\mathbf{E}'_h \in \mathbf{V}_h} \frac{\langle \mathbf{E}_h, \mathbf{E}'_h \rangle}{\|\mathbf{E}'_h\|_{\mathbf{L}^{q'}(\Omega)}} \ge \frac{1}{C} \|\mathbf{E}_h\|_{\mathbf{L}^q(\Omega)} \tag{4.19}$$

Furthermore,

$$\forall \psi_h \in Z_h^0, \sup_{\psi'_h \in Z_h^0} \frac{\langle \psi_h, \psi'_h \rangle}{\|\psi'_h\|_{L^2(\Omega)}} \ge \frac{1}{C} \|\psi_h\|_{L^2(\Omega)} \tag{4.20}$$

The following inf-sup condition follows:

$$\inf_{(\mathbf{E}_h,\psi_h) \in \mathbf{V}_h \times Z_h^0} \sup_{(\mathbf{E}'_h,\psi'_h) \in \mathbf{V}_h \times Z_h^0} \frac{\langle \mathbf{E}_h, \mathbf{E}'_h \rangle + \langle \psi_h, \psi'_h \rangle}{\left(\|\mathbf{E}_h\|_{\mathbf{L}^q(\Omega)} + \|\psi_h\|_{L^2(\Omega)}\right)\left(\|\mathbf{E}'_h\|_{\mathbf{L}^{q'}(\Omega)} + \|\psi'_h\|_{L^2(\Omega)}\right)} \ge \frac{1}{C} > 0 \tag{4.21}$$

This allows us to conclude that a verifies a uniform discrete inf-sup condition on $\mathcal{X}^q \times \mathcal{X}^{q'}$.

Furthermore $\mathbf{b}_{\phi_h}$ is a compact bilinear form and $H^1(\Omega) \ni \phi \mapsto \mathbf{b}_\phi$ is also compact.
From proposition 2.12, since $\phi_h(t)$ is in a bounded subset of $H^1(\Omega)$, we deduce that $\mathbf{a} + \mathbf{b}_{\phi_h}$ verifies a uniform discrete inf-sup condition independent of h and t.
We will now use duality estimates to deduce estimates for solutions $(\dot{\mathbf{E}}_h, \dot{\psi}_h, \beta_h)$ by applying proposition 2.13 to special spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_h$, $\mathcal{Y}_h$, $\mathcal{X}_+$, $\mathcal{Y}_-$.
We assume that

$$\begin{aligned}
\mathcal{X} &= \mathcal{X}^{q'} \\
\mathcal{Y} &= \mathcal{X}^q \\
\mathcal{X}_h &= \mathbf{Y}_h^1 \times Z_h^0 \times Y_h^0 = \mathcal{X}_h^{q'} \\
\mathcal{Y}_h &= \mathbf{Y}_h^1 \times Z_h^0 \times Y_h^0 = \mathcal{X}_h^q \\
\mathcal{X}_+ &= \mathbf{H}^{-1}(\Omega) \times H^{-1}(\Omega) \times W^{1,q'}(\Omega) \\
\mathcal{Y}_- &= \mathbf{H}_q(\operatorname{curl}, \Omega) \times H_0^1(\Omega) \times W_0^{1,q}(\Omega)
\end{aligned}$$

Let

$$\tilde{\mathbf{a}} = \mathbf{a} + \mathbf{b}_{\phi_h},$$

$v = (\mathbf{E}_0, \psi_0, \beta_0) \in \mathcal{X}'_+$ and $T'_h v = (\mathbf{E}'_h, \psi'_h, \beta'_h) \in \mathcal{Y}_h$ the solution of:

$$\tilde{\mathbf{a}}(u_h, T'_h v) = \langle u_h, v \rangle, \ \forall u_h \in \mathcal{X}_h$$

We denote any $u_h \in \mathcal{X}_h$ by $u_h = (\tilde{\mathbf{E}}_h, \tilde{\psi}_h, \tilde{\beta}_h)$. We have $T'_h v \in \mathcal{Y}_h$ so $T'_h v \in \mathcal{Y}_-$. We are looking for a bound on $T'_h v$ in the space $\mathcal{Y}_-$.

- A bound for β'_h in $W^{1,q}(\Omega)$ is given by the previous uniform discrete inf-sup condition on $\tilde{\mathbf{a}}$. One obtains:
$$\|\beta'_h\|_{W^{1,q}(\Omega)} \leq C \left(\|\mathbf{E}_0\|_{\mathbf{H}^1(\Omega)} + \|\psi_0\|_{H^1(\Omega)} + \|\beta_0\|_{W^{-1,q}(\Omega)} \right) \tag{4.22}$$
- One has:
$\langle \psi'_h, \tilde{\psi}_h \rangle - \langle \tilde{\psi}_h, \phi_h \beta'_h \rangle = \langle \psi_0, \tilde{\psi}_h \rangle, \ \forall \tilde{\psi}_h \in Z^0_h$.
The stability for P^0_h, the L^2 projection, in $H^{-1}(\Omega)$ and $q > 2$ give that:
$$\|\psi'_h\|_{H^1(\Omega)} \leq C(\|\phi_h \beta'_h\|_{H^1(\Omega)} + \|\psi_0\|_{H^1(\Omega)}) \leq C(\|\phi_h\|_{H^1(\Omega)} \|\beta'_h\|_{W^{1,q}(\Omega)} + \|\psi_0\|_{H^1(\Omega)})$$
Then using that $\|\phi_h\|_{L^\infty(0,T;H^1(\Omega))}$ is bounded independently of h and (4.22), we deduce:
$$\|\psi'_h\|_{H^1(\Omega)} \leq C(\|\mathbf{E}_0\|_{\mathbf{H}^1(\Omega)} + \|\psi_0\|_{H^1(\Omega)} + \|\beta_0\|_{W^{-1,q}(\Omega)})$$
- Furthermore
$\langle \mathbf{E}'_h, \tilde{\mathbf{E}}_h \rangle + \langle \tilde{\mathbf{E}}_h, \operatorname{grad} \beta'_h \rangle = \langle \mathbf{E}_0, \tilde{\mathbf{E}}_h \rangle = \langle P^1_h(\mathbf{E}_0), \tilde{\mathbf{E}}_h \rangle, \ \forall \tilde{\mathbf{E}}_h \in \mathbf{Y}^1_h$.

  We have the upper bound of $\mathbf{E}'_h$ in the L^q-norm by the inf-sup condition on $\tilde{\mathbf{a}}$. Concerning the L^2-norm of the curl of $\mathbf{E}'_h$:
Since $\mathbf{E}'_h + \operatorname{grad} \beta'_h = P^1_h(\mathbf{E}_0)$, we deduce that $\operatorname{curl} \mathbf{E}'_h = \operatorname{curl} P_h(\mathbf{E}_0)$. By the stability of the L^2 projection "from $\mathbf{H}^1$ to $\mathbf{H}(\operatorname{curl}, \Omega)$":
$$\|\operatorname{curl} \mathbf{E}'_h\|_{\mathbf{L}^2(\Omega)} \leq C \|\mathbf{E}_0\|_{\mathbf{H}^1(\Omega)}$$
And so $\|\mathbf{E}'_h\|_{\mathbf{L}^q(\Omega)} + \|\operatorname{curl} \mathbf{E}'_h\|_{\mathbf{L}^2(\Omega)} \leq C(\|\mathbf{E}_0\|_{\mathbf{H}^1(\Omega)} + \|\psi_0\|_{H^1(\Omega)} + \|\beta_0\|_{W^{-1,q}(\Omega)})$ where C doesn't depend on h.

To conclude

$$\|\psi'_h\|_{H^1(\Omega)} + \|\mathbf{E}'_h\|_{\mathbf{L}^q(\Omega)} + \|\operatorname{curl} \mathbf{E}'_h\|_{\mathbf{L}^2(\Omega)} + \|\beta'_h\|_{W^{1,q}(\Omega)} \leq C(\|\psi_0\|_{H^1(\Omega)} + \|\mathbf{E}_0\|_{\mathbf{H}^1(\Omega)} + \|\beta_0\|_{W^{-1,q}(\Omega)}) \tag{4.23}$$

which means that

$$\|T'_h v\|_{\mathcal{Y}_-} \leq \|v\|_{\mathcal{X}'_+} \tag{4.24}$$

Let l_h be the linear form given by:

$$l_h : \left\{ \begin{array}{lcl} \mathcal{Y}_- & \rightarrow & \mathbb{R} \\ (\mathbf{E}', \psi', \beta') & \mapsto & \langle \operatorname{curl} \mathbf{A}_h, \operatorname{curl} \mathbf{E}' \rangle + \langle D_{\mathbf{A}_h} \phi_h, \phi_h \mathbf{E}' \rangle + \langle D_{\mathbf{A}_h} \phi_h, D_{\mathbf{A}_h} \psi' \rangle \end{array} \right.$$

$l_h \in \mathcal{Y}'_-$, and $\|l_h\|_{L^\infty(0,T;\mathcal{Y}'_-)} \leq C$.

We can now use proposition 2.13 with $u = l_h$ to conclude that:

$$\|\dot{\mathbf{E}}_h\|_{L^\infty(0,T;\mathbf{H}^{-1}(\Omega))} + \|\dot{\psi}_h\|_{L^\infty(0,T;H^{-1}(\Omega))} + \|\beta_h\|_{L^\infty(0,T;W^{1,q'}(\Omega))} \leq C \tag{4.25}$$

□

4.5.2. *Conclusion on the convergence*

- E_h is bounded independently of h in $L^\infty(0,T;\mathbf{L}^2(\Omega))$, and $\dot{E}_h$ is bounded independently of h in $L^\infty(0,T;\mathbf{H}^{-1}(\Omega))$. It follows from theorem 2.4 that to extraction of a subsequence:

$$\mathbf{E}_h = -\dot{\mathbf{A}}_h \to -\dot{\mathbf{A}} \text{ in } \mathcal{C}(0,T;\mathbf{H}^{-s}(\Omega)) \text{ for all } 0 < s \leq 1$$

- A similar conclusion holds for ψ_h and ψ.

$$\psi_h = -\dot{\phi}_h \to -\dot{\phi} \text{ in } \mathcal{C}(0,T;H^{-s}(\Omega))\text{for all } 0 < s \leq 1$$

- Concerning the Lagrange multiplier β_h, one concludes that there exists $\beta \in L^\infty(0,T;W_0^{1,q'}(\Omega))$ such that:

$$\beta_h \underset{h\to 0}{\rightharpoonup} \beta \text{ in } L^\infty(0,T;W^{1,q'}(\Omega)) \text{ weak } *$$

  and

$$\beta_h \underset{h\to 0}{\rightharpoonup} \beta \text{ in } L^\infty(0,T;L^r(\Omega)) \text{ weak-* for all } r < \frac{2q'}{2-q'}.$$

4.6. The limit equation

We are now able to study the limit of equations (3.13)-(3.20). Convergence obtained on ϕ_h and $\mathbf{A}_h$ permits to take the limit on the right hand side of (3.15). The results obtained in section 4.5 leads to convergence on the left hand side. A weak convergence on β_h in the appropriate space is here sufficient due to the strong convergence obtained for ϕ_h. We then deduce that

$$\forall (\mathbf{E}',\psi',\beta') \in \mathcal{C}_c^\infty(]0,T[\times\Omega)^2 \times \mathcal{C}_c^\infty(]0,T[\times\Omega) \times \mathcal{C}_c^\infty([0,T]\times\Omega)$$

$$\int_0^T \langle \dot{\mathbf{E}},\mathbf{E}'\rangle dt + \int_0^T \langle \dot{\psi},\psi'\rangle dt + \int_0^T \langle \mathbf{E}',\operatorname{grad}\beta\rangle dt - \int_0^T \langle \psi',\phi\beta\rangle dt = \int_0^T \langle \operatorname{curl}\mathbf{A},\operatorname{curl}\mathbf{E}'\rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi,\phi\mathbf{E}'\rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi, D_{\mathbf{A}}\psi'\rangle dt \quad (4.26)$$

and

$$\int_0^T \langle \dot{\mathbf{E}},\operatorname{grad}\beta'\rangle dt - \int_0^T \langle \dot{\psi},\phi\beta'\rangle dt = 0 \quad (4.27)$$

Remark 4.14. *This formulation has a sense since we know from section 4.5.2 that* $\dot{\mathbf{E}} \in L^\infty(0,T;\mathbf{H}^{-1}(\Omega))$ *and* $\dot{\psi} \in L^\infty(0,T;H^{-1}(\Omega))$.

Remark 4.15. *Convergence on* $\mathbf{A}_h$ *and* ϕ_h *obtained in the 3D case (see remarks 4.9 and 4.12) prevent us to be able to pass to the limit in nonlinear terms on the right hand side.*

Remark 4.16. *From (4.27), one deduces that* $\operatorname{div}\mathbf{A} \in L^\infty(0,T;L^2(\Omega))$, *and in consequence* $\mathbf{A} \in L^\infty(0,T;\mathbf{H}^1(\Omega))$.

4.6.1. *Value of the Lagrange multiplier*

One can prove that β vanishes.

Let $\beta' \in \mathcal{C}_c^\infty(]0,T[\times\Omega)$.
Due to the regularity in time of the solution, this formulation is also valid almost everywhere on $[0,T]$. We then apply the almost everywhere version of (4.26) to test functions $\mathbf{E}' = \operatorname{grad}\beta'$ and $\psi' = -\phi\beta'$ and obtain using (4.27):

$$\langle \operatorname{grad}\beta',\operatorname{grad}\beta\rangle + \langle \phi\beta',\phi\beta\rangle = 0. \quad (4.28)$$

Since $\beta \in L^\infty(0,T;W_0^{1,q'}(\Omega))$, $|\phi|^2\beta \in L^\infty(0,T;L^2(\Omega))$, one then deduce by regularity of solutions of elliptic equations that $\beta \in H_0^1(\Omega)$. This implies

$$\langle \operatorname{grad}\beta, \operatorname{grad}\beta\rangle + \langle \phi\beta, \phi\beta\rangle = 0. \tag{4.29}$$

And consequently, $\beta \equiv 0$.

4.6.2. *Weak solution of Maxwell Klein Gordon equation*

One concludes that $\forall(\mathbf{E}', \psi') \in \mathcal{C}_c^\infty(]0,T[\times\Omega)^2 \times \mathcal{C}_c^\infty(]0,T[\times\Omega)$:

$$-\int_0^T \langle \mathbf{E}, \dot{\mathbf{E}}'\rangle dt - \int_0^T \langle \psi, \dot{\psi}'\rangle dt = \int_0^T \langle \operatorname{curl}\mathbf{A}, \operatorname{curl}\mathbf{E}'\rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi, \phi\mathbf{E}'\rangle dt + \int_0^T \langle D_{\mathbf{A}}\phi, D_{\mathbf{A}}\psi'\rangle dt \tag{4.30}$$

$(\mathbf{E}, \psi)$ is then a weak solution of Maxwell Klein Gordon equation in the sense of definition 3.2. This completes the proof of theorem 4.1.

4.6.3. *Uniqueness of a weak solution*

The solution $(\mathbf{E}, \mathbf{A}, \psi, \phi)$ obtained verifies equation (4.30), and due to its regularity, following lemma 8.2. from [21], one obtains:

$$(\mathbf{E}, \mathbf{A}, \psi, \phi) \in \mathcal{C}_w(0,T;\mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)) \tag{4.31}$$

Now using the result in [22], one deduces the uniqueness of this solution and moreover that:

$$(\mathbf{E}, \mathbf{A}, \psi, \phi) \in \mathcal{C}(0,T;\mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)) \tag{4.32}$$

5. Conclusion

We have proved that the constraint preserving scheme converges to a weak solution of the Maxwell-Klein-Gordon equation.
This result leads also to a result of existence of solution with data of finite energy. Unfortunately the proof of convergence does not extend to the tridimensional case (due to the default of compactness of the Sobolev imbedding), as pointed out by the corresponding remarks throughout the paper. But this problem could be investigated in a further work using the notion *concentration compactness*.

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